

Differential-geometrical approach to the dynamics of dissipationless incompressible Hall magnetohydrodynamics: I. Lagrangian mechanics on semidirect product of two volume preserving diffeomorphisms and conservation laws

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Abstract

The dynamics of a dissipationless incompressible Hall magnetohydrodynamic (HMHD) medium is formulated using Lagrangian mechanics on a semidirect product of two volume preserving diffeomorphism groups. In the case of \mathbb{T}^3 or E^3 , the generalized Elsässer variables introduced by Galtier (S. Galtier 2006 J. Plasma Phys. **72** 721-69) yield remarkably simple expressions of basic formulas and equations such as the structure constants of Lie algebra, the equation of motion, and the conservation laws. *Four* constants of motion, where three of the four are independent, are naturally derived from the generalized Elsässer variables representation of the equation of motion for the HMHD system: total plasma energy, magnetic helicity, hybrid helicity, and the modified cross helicity.

1 Introduction

About a half century ago, when Arnold reviewed his study on the dynamical systems on Lie groups and related hydrodynamic topics in a unified form, he treated three materials in somewhat intertwining manner (Arnold 1966), these were: (1) Lagrangian mechanics on Lie groups and derivation of the equation of motion including exposition of its relation to Noether's theory, (2) Hamiltonian mechanics on Lie groups and the consideration of stability problems around stationary solutions, and (3) reinterpretation of the equation of motion as a geodesic equation under an appropriate Affine connection on a Lie group and its application to predictability problems in terms of the Riemannian curvature

induced by the connection. Since then, these three topics have evolved separately. Today, the Hamiltonian mechanics on a wide varieties of Lie groups are summarized in terms of the Lie-Poisson structure of the cotangent bundle of Lie groups (Marsden and Ratiu 1994, Morrison 1998 for comprehensive reviews) and the Euler-Poincare structure has been established as its Lagrangian mechanical counterpart (Holm Marsden and Ratiu 1998 for a comprehensive review) and a number of dynamical systems have been reformulated according to these approaches.

In the present study, we treat Hall magnetohydrodynamic (HMHD) systems from a Lagrangian mechanical viewpoint. HMHD systems are regarded as one of the simplest models of a plasma that includes the so-called two fluid effect, i.e., the effect arising from the difference between the ion and electron current field (Lighthill 1960). In HMHD formulation, Ohm's law is approximated as follows:

$$\mathbf{E} + \frac{1}{c} \mathbf{u} \times \mathbf{B} = \frac{1}{qnc} \mathbf{j} \times \mathbf{B}, \quad (1)$$

where \mathbf{E} , c , \mathbf{u} , \mathbf{B} , q , n , \mathbf{j} are electric field, the speed of light, mean velocity of the plasma, magnetic field, electric charge, number density of ions, and current field, respectively. The Hall effect term on the right hand side reflects the influence of the finite size of the ion skin depth. HMHD systems have been investigated intensively in the context of both space and fusion plasmas.

The effects of the Hall term on the properties of magnetohydrodynamic (MHD) turbulence have been investigated from various viewpoints including the closure approach in the weak/wave turbulence framework (Galtier 2006), evaluation of turbulent energy transfer and dynamo action using direct numerical simulation (DNS) data (Mininni et al. 2007), and coherent structure formation by DNS (Miura and Araki 2014).

From a mathematical perspective, HMHD systems have gradually attracted the attention of researchers for such topics as the conservation laws of Lagrangian invariant form (Shivamoggi 2009, Araki 2009), existence of a solution to initial value problem (Acheritogaray et al. 2011, Chae et al. 2014). In terms of the analytical mechanical studies of HMHD systems, since Holm's pioneering work in this field (Holm 1987), Hamiltonian approaches have predominated (Sahraoui et al. 2003, Hirota et al. 2006, Yoshida and Hameiri 2013). Keramidas Charidakos et al. did however employ the Lagrangian approach to derive the HMHD momentum equation and Ohm's law using the first variation of action constructed from a general two-fluid plasma Lagrangian (Keramidas Charidakos et al. 2014). To the author's knowledge, little research has focused on the Lagrangian mechanical approach associated with the geodesic/the Riemannian curvature formulation which latter originates from Arnold's third formulation, although much literature exists regarding standard MHD system (Zeitlin and Kambe 1993, Hattori Y 1994). In this study, we examine HMHD system from Arnold's first formulation in the 1966 article. We will present a mathematical model that derives the HMHD equations as Euler-Lagrange equations for variational problems and examine its configuration space, described using a Lie

group.

In the present study, we focus on the three-dimensional case and do not treat the more general N -dimensional case for the sake of mathematical simplicity.

This paper is organized as follows: in section 2, the foundations of Lagrangian mechanics is discussed and the equation of motion derived from Hamilton's principle. In section 3, we discuss the Lie algebraic structure of the function space of the velocity and magnetic fields. The equation of motion, expanded by the generalized Elsässer variables (GEV), is reviewed from a differential geometrical viewpoint in Section 4. Section 5 is devoted to the derivation of the constants of motion from the GEV derived equation of motion. The summary is presented in section 6.

2 Configuration space and Lagrangian mechanics of HMHD systems

Although a number of continuum systems have been investigated using semidirect product groups, most of them are “passive” to the second element of the product, i.e.,

$$(g_1, v_1) \circ (g_2, v_2) = (g_1 \circ g_2, v_1 + B(g_1)v_2), \quad (2)$$

where $g_1, g_2 \in G$, $v_1, v_2 \in V$, and $B: G \times V \rightarrow V$ is an appropriate representation of G on a vector space V (Marsden and Ratiu 1994, Arnold and Khesin 1998). An exception is the case treated by Vizman (2001), in which the product is given by

$$(g_1, h_1) \circ (g_2, h_2) = (g_1 \circ g_2, h_1 \circ B(g_1)h_2), \quad (3)$$

where h_1, h_2 are the elements of a Lie group H , $B: G \times H \rightarrow H$ is group homomorphism. As we will demonstrate, the dynamics of an HMHD system is founded on this group action and induced Lie algebra so that we will follow the work of Vizman in the present study.

Let M be a container of an HMHD fluid that is a three-dimensional differentiable manifold with coordinate system (x^i) . The basic variables of an HMHD system in this study are a pair of volume preserving diffeomorphism maps on M denoted by (\vec{X}, \vec{Y}) where $\vec{X}, \vec{Y} \in S\text{Diff}(M)$.¹

The first element $\vec{X} = \vec{X}(\vec{x}, t) = (X^i(\vec{x}, t))$ is a triplet of functions on M . The value $\vec{X}(\vec{a}, t)$ physically implies the position of a fluid particle at time t , which was initially located at $\vec{a} \in M$. The map \vec{X} is related to the velocity field

¹In this paper, we use arrow symbol on top of a multifunctional character of mathematical quantities. For example, diffeomorphism (a triplet of functions) is expressed by $\vec{X} = (X^1, X^2, X^3)$, and a pair of vector fields by $\vec{Z} = (\mathbf{u}, \mathbf{b})$. Boldface letters are used to denote vector fields on M .

in the Eulerian specification $\mathbf{u}(t) = u^i(t) \frac{\partial}{\partial x^i} \in \mathfrak{X}_\Sigma(M)$ by the formula

$$\left. \frac{\partial X^i(\vec{a}, \tau)}{\partial \tau} \right|_{\tau=t} \left(\frac{\partial}{\partial x^i} \right)_{\vec{x}=\vec{X}(\vec{a}, t)} = u^i \left(\vec{X}(\vec{a}, t), t \right) \left(\frac{\partial}{\partial x^i} \right)_{\vec{x}=\vec{X}(\vec{a}, t)},$$

where $\mathfrak{X}_\Sigma(M)$ is the Lie algebra of $S\text{Diff}(M)$, i.e., the function space of divergence-free vector fields equipped with Lie bracket $[\mathbf{a}, \mathbf{b}]$. We denote the space of \vec{X} by G hereafter.

The second element $\vec{Y} = \vec{Y}(\vec{a}, s; t) = (Y^i(\vec{a}, s; t))$ is a “stream function” of the current field where s and t are the current line parameter and time, respectively. The map \vec{Y} is related to the current field in the Eulerian specification $\mathbf{j}(t) = j^i(t) \frac{\partial}{\partial x^i} \in \mathfrak{X}_\Sigma(M)$ by the formula

$$\frac{\partial Y^i(\vec{a}, s; t)}{\partial s} \left(\frac{\partial}{\partial x^i} \right)_{\vec{x}=\vec{Y}(\vec{a}, s; t)} = -\alpha j^i \left(\vec{Y}(\vec{a}, s; t), t \right) \left(\frac{\partial}{\partial x^i} \right)_{\vec{x}=\vec{Y}(\vec{a}, s; t)},$$

where α is a parameter that determines the relative strength of the Hall term. We denote the space of \vec{Y} by H hereafter.

Group operation of a diffeomorphism, $S\text{Diff}(M)$, is given by the composition of the function triplets and we denote it by $\vec{X}_1 \circ \vec{X}_2 = \vec{X}_1(\vec{X}_2)$. By defining the group operation for the pairs of basic variables (\vec{X}, \vec{Y}) by the function compositions

$$(\vec{X}_1, \vec{Y}_1) \circ (\vec{X}_2, \vec{Y}_2) = (\vec{X}_1(\vec{X}_2), \vec{Y}_1(\vec{X}_1(\vec{Y}_2(\vec{X}_1^{-1})))) ,$$

we obtain the semidirect product of the diffeomorphisms $G \ltimes H$.

Substituting an exponential map approximation for each variable $\vec{X}_k(\tau) = \exp(\tau \mathbf{u}_k)$ and $\vec{Y}_k(\tau; t) = \exp(-\tau \alpha \mathbf{j}_k(t))$, using Hausdorff’s formula $\exp(\mathbf{a}) \exp(\mathbf{b}) = \exp(\mathbf{a} + \mathbf{b} + \frac{1}{2}[\mathbf{a}, \mathbf{b}] + \dots)$ and taking $O(\tau^2)$ terms of function compositions, we obtain the following Lie bracket on $\mathfrak{g} \ltimes \mathfrak{h}$, which defines the Lie algebra of $G \ltimes H$:

$$\begin{aligned} [\vec{\mathbf{V}}_1, \vec{\mathbf{V}}_2] = & \left(\nabla \times (\mathbf{u}_1 \times \mathbf{u}_2), \right. \\ & \left. -\alpha \left(\nabla \times (\mathbf{u}_1 \times \mathbf{j}_2) + \nabla \times (\mathbf{j}_1 \times \mathbf{u}_2) - \alpha \nabla \times (\mathbf{j}_1 \times \mathbf{j}_2) \right) \right), \end{aligned} \quad (4)$$

where $\vec{\mathbf{V}}_k := (\mathbf{u}_k, -\alpha \mathbf{j}_k)$ is a tangent vector of one parameter subgroup $\gamma_k(\tau) := (\vec{X}_k(\tau), \vec{Y}_k(\tau; t))$, i.e., a small path on the configuration space $G \ltimes H$, and $\nabla \times$ is the curl operator in standard vector analysis notation. Since M is three-dimensional and the vector fields considered here are divergence free, the Lie bracket on $\mathfrak{X}_\Sigma(M)$ is given by $[\mathbf{a}, \mathbf{b}] = \nabla \times (\mathbf{a} \times \mathbf{b})$. Note that, if the semidirect product operation (2) is used, the current fields product term in (4) does not appear in the Lie bracket, and thus, the evolution equation of standard one-fluid MHD is obtained (Hattori 1994).

Let $\gamma(t)$ ($t \in [0, 1]$) be a path on $G \ltimes H$ on which a sufficiently short interval between the two points $\gamma(t)$ and $\gamma(t + \tau)$ is well approximated by $(\exp(\tau \mathbf{u}(t)),$

$\exp(-\tau\alpha\mathbf{j}(t))$). To consider the variational problem along this path, we exert a perturbation on the path $\gamma(t; \delta) = (\exp(\delta\boldsymbol{\xi}(t)), \exp(-\delta\alpha\boldsymbol{\eta}(t))) \circ \gamma(t; 0)$, where δ is a small parameter, $\gamma(t; 0) = \gamma(t)$, and $\boldsymbol{\xi}, \boldsymbol{\eta}$ are the “displacement” fields. Let $\vec{\mathbf{V}}(t; \delta) = \vec{\mathbf{V}}(t) + \delta\vec{\mathbf{v}}(t) + o(\delta)$ be the tangent vector along the perturbed path where $\vec{\mathbf{v}}(t)$ is the perturbation part of the tangent vector. For the case of the dynamical systems on Lie groups, the perturbation part $\vec{\mathbf{v}}(t)$ is known to satisfy Lin’s constraint (Cendra and Marsden 1987)

$$\vec{\mathbf{v}} = \dot{\vec{\mathbf{v}}} + [\vec{\mathbf{v}}, \vec{\mathbf{V}}], \quad (5)$$

where $\vec{\mathbf{v}} := (\boldsymbol{\xi}(t), -\alpha\boldsymbol{\eta}(t))$.

To define the Riemannian metric on $G \ltimes H$, we introduce the inverse of the curl operation, which corresponds physically to the calculation of the magnetic field \mathbf{b} induced by a current field \mathbf{j} . Hereafter, $(\nabla \times)^{-1}$ denotes the operation to obtain the vector potential \mathbf{b} from a divergence-free field \mathbf{j} that satisfies

$$\mathbf{j} = \nabla \times \mathbf{b}, \quad \nabla \cdot \mathbf{b} = 0, \quad \mathbf{b}^{(H)} = \mathbf{0}, \quad (6)$$

where $\nabla \cdot$ is the divergence operator and $\mathbf{b}^{(H)}$ is harmonic function component of Hodge-Kodaira decomposition of \mathbf{b} (Yoshida and Giga 1990).

Here we define Riemannian metric on $G \ltimes H$ by

$$\begin{aligned} \langle \vec{\mathbf{V}}_1 | \vec{\mathbf{V}}_2 \rangle_{(\vec{X}, \vec{Y})} &:= \int_{\vec{a} \in M} (\mathbf{u}_1 \cdot \mathbf{u}_2)_{\vec{X}(\vec{a}, t)} d^3 \vec{X}(\vec{a}, t) \\ &+ \int_{\vec{b} \in M} \left(((\nabla \times)^{-1} \mathbf{j}_1) \cdot ((\nabla \times)^{-1} \mathbf{j}_2) \right)_{\vec{Y}(\vec{b}, s; t)} d^3 \vec{Y}(\vec{b}, s; t), \end{aligned} \quad (7)$$

where $(\vec{X}, \vec{Y}) \in G \ltimes H$ and $d^3 \vec{X}(\vec{a}, t)$, $d^3 \vec{Y}(\vec{b}, s; t)$ are the advected volume elements initially located at \vec{a} and \vec{b} , respectively. Since the maps \vec{X} and \vec{Y} are volume preserving: $d^3 \vec{x} = d^3 \vec{X}(\vec{x}, t) = d^3 \vec{Y}(\vec{x}, s; t)$, the value of the Riemannian metric is invariant against the right-translation. We define the Lagrangian

$$L = \frac{1}{2} \langle \vec{\mathbf{V}} | \vec{\mathbf{V}} \rangle_{(\vec{X}, \vec{Y})} \quad (8)$$

to be the total plasma energy. Note that this Lagrangian is the same as that used in Hattori (2011) to derive the equation of motion of an incompressible standard MHD fluid. The first variation of action associated with this Lagrangian becomes

$$\begin{aligned} \frac{dS}{d\delta} &= \int_0^1 \langle \vec{\mathbf{V}} | \vec{\mathbf{v}} \rangle_{(\vec{X}, \vec{Y})} dt = \int_0^1 \langle \vec{\mathbf{V}} | \vec{\mathbf{v}} \rangle_{(e, e)} dt = \int_0^1 \langle \vec{\mathbf{V}} | \dot{\vec{\mathbf{v}}} + [\vec{\mathbf{v}}, \vec{\mathbf{V}}] \rangle_{(e, e)} dt \\ &= \int_0^1 dt \int_{\vec{x} \in M} d^3 \vec{x} \left[\mathbf{u} \cdot (\boldsymbol{\xi} + \nabla \times (\boldsymbol{\xi} \times \mathbf{u})) \right. \\ &\quad \left. + \mathbf{b} \cdot (\dot{\boldsymbol{\beta}} + (\boldsymbol{\xi} \times \mathbf{j}) + (\boldsymbol{\eta} \times \mathbf{u}) - \alpha(\boldsymbol{\eta} \times \mathbf{j}))_S \right], \end{aligned} \quad (9)$$

where e is the identity map, $\mathbf{b} := (\nabla \times)^{-1} \mathbf{j}$, $\boldsymbol{\beta} := (\nabla \times)^{-1} \boldsymbol{\eta}$ and hereafter $(*)_S$ is the projection onto the solenoidal component of a vector field. Integration by parts with fixed end conditions yields the following Euler-Lagrange equation:

$$\dot{\mathbf{u}} = (\mathbf{u} \times (\nabla \times \mathbf{u}) + \mathbf{j} \times \mathbf{b})_S, \quad (10)$$

$$\dot{\mathbf{b}} = \nabla \times (\mathbf{u} \times \mathbf{b}) - \alpha \nabla \times (\mathbf{j} \times \mathbf{b}). \quad (11)$$

Thus, the evolution equation of an incompressible ideal Hall magnetohydrodynamic medium is derived as a dynamical system on a semidirect product of two diffeomorphism groups. Note that in the limit $\alpha \rightarrow 0$, we obtain the standard MHD model for incompressible plasmas. For the purely hydrodynamic case ($\mathbf{j} = \mathbf{b} = \mathbf{0}$), the Euler equation for an incompressible fluid is obtained.

As is obvious from our formulation, by setting the Hall strength parameter to $\alpha = 0$ in (4), one simply obtains the Euler equation. The second element of $\vec{\mathbf{V}}$ -variables should be taken as such a vector field that needs to be divided by α to obtain $O(1)$ vector field. Thus, the standard MHD system should be considered as a kind of singular limit of present formulation. Analytical mechanical approaches to the HMHD systems seem to raise such small parameter problem when its relation to the standard MHD limit is considered. For example, in the Hamiltonian mechanics approach, one of the natural choices of vector variables is the pair of the total ion momentum density $\mathbf{M} = \rho \mathbf{v} + R^{-1} a \rho \mathbf{A}$ and the magnetic vector potential \mathbf{A} , where $R/a = \alpha$ in our notation (Holm 1987). In the limit $\alpha \rightarrow 0$, these two variables come close to each other $\mathbf{M} \approx R^{-1} a \rho \mathbf{A}$, and manipulation of the small difference $\mathbf{v} = \mathbf{M}/\rho - R^{-1} a \mathbf{A}$ is needed to capture the ion flow.

3 Lie algebra induced on the space of (\mathbf{u}, \mathbf{b})

For practical purposes, it is convenient to use the magnetic field \mathbf{b} instead of the current field \mathbf{j} . Here, we introduce a pair of velocity and magnetic fields by $\vec{\mathbf{Z}} = (\mathbf{u}, \mathbf{b})$. From the variational calculation process (9), the bilinear operation on the $\vec{\mathbf{Z}}$ -variable space defined by the following formula is expected to work as a commutator of certain Lie algebra:

$$\begin{aligned} \{\vec{\mathbf{Z}}_1, \vec{\mathbf{Z}}_2\} = & \left(\nabla \times (\mathbf{u}_1 \times \mathbf{u}_2), \right. \\ & \left. (\mathbf{u}_1 \times (\nabla \times \mathbf{b}_2) + (\nabla \times \mathbf{b}_1) \times \mathbf{u}_2 - \alpha (\nabla \times \mathbf{b}_1) \times (\nabla \times \mathbf{b}_2))_S \right), \end{aligned} \quad (12)$$

where $\vec{\mathbf{Z}}_k := (\mathbf{u}_k, \mathbf{b}_k)$. The definition (12) is skew symmetric. Introducing the inner product on the $\vec{\mathbf{Z}}$ -variable space given by

$$\langle \vec{\mathbf{Z}}_1 | \vec{\mathbf{Z}}_2 \rangle_Z := \int_{\vec{x} \in M} (\mathbf{u}_1 \cdot \mathbf{u}_2 + \mathbf{b}_1 \cdot \mathbf{b}_2) d^3 \vec{x}, \quad (13)$$

we see that the value of the Lagrangian (8) and the value of the inner product of the commutator and a variable are preserved under this mapping:

$$L = \frac{1}{2} \langle \vec{V} | \vec{V} \rangle_{(e,e)} = \frac{1}{2} \langle \vec{Z} | \vec{Z} \rangle_Z, \quad (14)$$

$$\langle \vec{V}_1 | [\vec{V}_2, \vec{V}_3] \rangle_{(e,e)} = \langle \vec{Z}_1 | \{ \vec{Z}_2, \vec{Z}_3 \} \rangle_Z, \quad (15)$$

where \vec{Z}_k ($k = 1, 2, 3$) are induced from $\vec{V}_k = (\mathbf{u}_k, -\alpha \mathbf{j}_k)$ by $\mathbf{b}_k = (\nabla \times)^{-1} \mathbf{j}_k$. It is straightforward but lengthy to check that the definition (12) satisfies the Jacobi identity. The commutator defines a Lie algebra on \vec{Z} -variable space.

Under this mapping from \vec{V} -variable space to \vec{Z} -variable space, the variational problem is also formulated on \vec{Z} -variable space. The first variation of action is mapped as follows:

$$\int_0^1 \langle \vec{V} | \dot{\vec{v}} + [\vec{v}, \vec{V}] \rangle_{(e,e)} dt = \int_0^1 \langle \vec{Z} | \dot{\vec{\zeta}} + \{ \vec{\zeta}, \vec{Z} \} \rangle_Z dt, \quad (16)$$

where $\vec{\zeta}(t) := (\boldsymbol{\xi}(t), \boldsymbol{\beta}(t))$ is the displacement field in \vec{Z} -variable space which is mapped from $\vec{v}(t) = (\boldsymbol{\xi}(t), -\alpha \boldsymbol{\eta}(t))$. Thus, the Lie algebraic structure on \vec{Z} -variable space is also expected to work as a framework of the Lagrangian mechanics of an ideal incompressible HMHD medium. In the following, we treat the HMHD system as a dynamical system on the Lie algebra of \vec{Z} -variable space.

4 Generalized Elsässer variables representation

In this section, we demonstrate that in the case of $M = \mathbb{T}^3$ or E^3 , the *generalized Elsässer variables* (GEV) (Galtier 2006) yield remarkably simple expressions of basic formulas or equations such as the structure constant of Lie algebra, equation of motion, etc. In this section, we review Galtier's derivation from differential geometrical viewpoint.

Since the velocity and magnetic fields satisfy divergence-free conditions $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{b} = 0$, they can be expanded in the complex helical wave (CHW) modes as

$$\mathbf{u}(\vec{x}, t) = \sum_{\vec{k}, \sigma_k} \hat{u}(\vec{k}, \sigma_k; t) \phi(\vec{k}, \sigma_k; \vec{x}), \quad \mathbf{b}(\vec{x}, t) = \sum_{\vec{k}, \sigma_k} \hat{b}(\vec{k}, \sigma_k; t) \phi(\vec{k}, \sigma_k; \vec{x}), \quad (17)$$

where a circumflex denotes a Fourier coefficient and the function $\phi(\vec{k}, \sigma_k; \vec{x})$ is a normalized complex helical wave (CHW). A CHW is defined by

$$\phi(\vec{k}, \sigma_k; \vec{x}) := 2^{-\frac{1}{2}} \mathbf{h}(\vec{k}, \sigma_k) e^{2\pi i \vec{k} \cdot \vec{x}}, \quad (18)$$

where

$$\mathbf{h}(\vec{k}, \sigma_k) := \mathbf{e}_\theta(\vec{k}) + i \sigma_k \mathbf{e}_\phi(\vec{k}), \quad (19)$$

σ_k , \vec{k} , \mathbf{e}_θ , \mathbf{e}_ϕ are the helical vector, helicity ($\sigma_k = \pm 1$), wavenumber, and the base vector of spherical coordinate system on wavenumber space in the θ - and ϕ -directions, respectively (Lesieur 1997 Sect.V-5, Waleffe 1992).

To derive the GEV representation of an HMHD system, we introduce the base functions of \vec{Z} -variable space, each of which is given by

$$\vec{\Phi}(\vec{k}, \sigma_k, s_k; \vec{x}) = \begin{pmatrix} \phi(\vec{k}, \sigma_k; \vec{x}) \\ \lambda_{\sigma_k}^{s_k}(k) \phi(\vec{k}, \sigma_k; \vec{x}) \end{pmatrix}, \quad (20)$$

where $s_k = \pm 1$, $k = |\vec{k}|$, and $\lambda_{\sigma_k}^{s_k}(k)$ is given by

$$\lambda_{\sigma_k}^{s_k}(k) = \sigma_k \left(s_k \sqrt{(\pi \alpha k)^2 + 1} - \pi \alpha k \right). \quad (21)$$

The functions $\vec{\Phi}$ are derived as eigenfunctions of the linear problem with a uniform background magnetic field \mathbf{B}_0 : $\partial_t \vec{Z} = \hat{L} \vec{Z}$, where

$$\hat{L} \vec{Z} = \begin{pmatrix} (\mathbf{j} \times \mathbf{B}_0)_S \\ \nabla \times ((\mathbf{u} - \alpha \mathbf{j}) \times \mathbf{B}_0) \end{pmatrix}. \quad (22)$$

In terms of the complex helical wave mode representation (17), the linear operator \hat{L} can be reduced to $\hat{L} = 2\pi i B_0 k_\parallel \hat{M}$, where

$$\hat{M} = \begin{pmatrix} O & I \\ I & -\alpha \nabla \times \end{pmatrix} = \begin{pmatrix} O & I \\ I & -2\pi \alpha \sigma_k k I \end{pmatrix}, \quad (23)$$

$B_0 = |\mathbf{B}_0|$, $k_\parallel = \vec{k} \cdot (\mathbf{B}_0/B_0)$ and I is the identity operator. The eigenequation of \hat{M} is $\lambda^2 + 2\pi \alpha \sigma_k k \lambda - 1 = 0$ and the solutions $\lambda_{\sigma_k}^{s_k}(k)$ satisfy the following relations:

$$\lambda_{\sigma_k}^{s_k}(k) + \lambda_{\sigma_k}^{-s_k}(k) = -2\pi \alpha \sigma_k k, \quad (24)$$

$$\lambda_{\sigma_k}^{s_k}(k) \lambda_{\sigma_k}^{-s_k}(k) = -1. \quad (25)$$

The eigenvalues and eigenfunctions of the operator \hat{M} are given by Eqs.(21) and (20), respectively. The eigenfunctions physically represent the ion cyclotron waves ($s_k = 1$ for left-hand circular polarization) and the whistler waves ($s_k = -1$ for right-hand circular polarization), respectively (Galtier 2006, Cramer 2011 §2.3.1).

Since the eigenvalues (21) are real and non-degenerate, $\vec{\Phi}(\vec{k}, \sigma_k, s_k; \vec{x})$ are orthogonal:

$$\left\langle \overline{\vec{\Phi}(\vec{k}, \sigma_k, s_k; \vec{x})} \middle| \vec{\Phi}(\vec{p}, \sigma_p, s_p; \vec{x}) \right\rangle_Z = (1 + (\lambda_{\sigma_k}^{s_k}(k))^2) \delta_{(\vec{k}, \sigma_k, s_k), (\vec{p}, \sigma_p, s_p)}^5, \quad (26)$$

where an overline denotes a complex conjugate and δ^5 is a product of Kronecker's deltas. Note that, this formula gives component of the Riemannian metric tensor.

Since the eigenfunctions themselves do not depend on the intensity of \mathbf{B}_0 , they work as orthogonal bases of an HMHD system even if the uniform background magnetic field is absent. We introduce here the expression²:

$$\vec{\mathbf{Z}}(\vec{x}, t) = \sum_{\vec{k}} \vec{\mathbf{Z}}(\vec{k}; \vec{x}, t), \quad (27)$$

where the notation $\vec{k} := (\vec{k}, \sigma_k, s_k)$ is introduced for brevity. The component $\vec{\mathbf{Z}}(\vec{k})$ and corresponding Fourier coefficient $\hat{Z}(\vec{k})$ are given by

$$\vec{\mathbf{Z}}(\vec{k}; \vec{x}, t) := \hat{Z}(\vec{k}; t) \vec{\Phi}(\vec{k}; \vec{x}), \quad (28)$$

$$\hat{Z}(\vec{k}; t) := \left\langle \vec{\Phi}(\vec{k}; \vec{x}) \middle| \vec{\mathbf{Z}}(\vec{x}, t) \right\rangle_Z / \left(1 + (\lambda_{\sigma_k}^{s_k}(k))^2 \right), \quad (29)$$

respectively. The relation between the complex helical Fourier coefficients of velocity and magnetic fields $\hat{u}(\vec{k}, \sigma_k)$, $\hat{b}(\vec{k}, \sigma_k)$ and the generalized Elsässer variable $\hat{Z}(\vec{k})$ is given by

$$\hat{Z}(\vec{k}; t) = \frac{\hat{u}(\vec{k}; t) + \lambda_{\sigma_k}^{s_k}(k) \hat{b}(\vec{k}; t)}{1 + (\lambda_{\sigma_k}^{s_k}(k))^2}, \quad (30)$$

where the notation $\hat{k} := (\vec{k}, \sigma_k)$ is introduced for brevity. The velocity and magnetic fields are expressed in terms of $\hat{Z}(\vec{k})$ as follows:

$$\mathbf{u}(\vec{x}, t) = \sum_{\vec{k}} \hat{Z}(\vec{k}; t) \phi(\hat{k}; \vec{x}), \quad (31)$$

$$\mathbf{b}(\vec{x}, t) = \sum_{\vec{k}} \lambda_{\sigma_k}^{s_k}(k) \hat{Z}(\vec{k}; t) \phi(\hat{k}; \vec{x}). \quad (32)$$

Substituting the mode expansions (31) and (32) into Eq.(12) and taking the inner product with $\vec{\mathbf{Z}}(\vec{k}; \vec{x})$, we obtain

$$\left\langle \vec{\mathbf{Z}}(\vec{k}) \middle| \left\{ \vec{\mathbf{Z}}(\vec{p}), \vec{\mathbf{Z}}(\vec{q}) \right\} \right\rangle_Z = ((\vec{k} \parallel \vec{p} \parallel \vec{q})) \lambda_{\sigma_k}^{-s_k}(k) \hat{Z}(\vec{k}) \hat{Z}(\vec{p}) \hat{Z}(\vec{q}), \quad (33)$$

where the symbol $((\vec{k} \parallel \vec{p} \parallel \vec{q}))$ is defined by the integral of the scalar triple product of the complex helical waves as follows³:

$$((\vec{k} \parallel \vec{p} \parallel \vec{q})) := \alpha^{-1} (\hat{k} | \hat{p} | \hat{q}) \left(\left(\lambda_{\sigma_k}^{s_k}(k) \lambda_{\sigma_p}^{s_p}(p) \lambda_{\sigma_q}^{s_q}(q) \right)^2 - 1 \right), \quad (34)$$

$$(\hat{k} | \hat{p} | \hat{q}) := \int_{\vec{x} \in M} \phi(\hat{k}; \vec{x}) \cdot \left(\phi(\hat{p}; \vec{x}) \times \phi(\hat{q}; \vec{x}) \right) d^3 \vec{x}. \quad (35)$$

²Note that, since $\lambda_{\sigma_k}^{s_k}(k) \approx \sigma_k s_k - \sigma_k (\pi \alpha k) + \frac{1}{2} \sigma_k s_k (\pi \alpha k)^2 + O(\alpha^4)$, the mode sums $\hat{Z}(\vec{k}, +, +) \phi(\vec{k}, +) + \hat{Z}(\vec{k}, -, -) \phi(\vec{k}, -)$, $\hat{Z}(\vec{k}, +, -) \phi(\vec{k}, +) + \hat{Z}(\vec{k}, -, +) \phi(\vec{k}, -)$ converge to the Elsässer variables of a standard MHD system $\mathbf{z}_+ = \mathbf{u} + \mathbf{b}$, $\mathbf{z}_- = \mathbf{u} - \mathbf{b}$ in the limit $\alpha \rightarrow 0$.

The derivation is given in Appendix 1. The symbols $((* \| * \| *))$ and $(* | * | *)$ have the following properties: (1) they have skew and cyclic symmetry under the permutations of mode indices; (2) they are non zero if the wavenumbers constitute a closed triangle $\vec{k} + \vec{p} + \vec{q} = \vec{0}$; and (3) they are zero if the wavenumbers are parallel $\vec{p} \parallel \vec{q}$. We obtain \hat{Z} -representation of the Lie bracket, i.e., the structure constants of Lie algebra for the generalized Elsässer variables are as follows:

$$\begin{aligned} \{ \vec{Z}(\vec{p}), \vec{Z}(\vec{q}) \} &= \sum_{\sigma_k, s_k} ((-\vec{p} - \vec{q}, \sigma_k, s_k \| \vec{p} \| \vec{q})) \\ &\times \frac{\lambda_{\sigma_k}^{-s_k}(|\vec{p} + \vec{q}|)}{1 + (\lambda_{\sigma_k}^{-s_k}(|\vec{p} + \vec{q}|))^2} \hat{Z}(\vec{p}) \hat{Z}(\vec{q}) \vec{\Phi}(\vec{p} + \vec{q}, \sigma_k, s_k; \vec{x}). \end{aligned} \quad (38)$$

Applying this formula to the expression for the first variation (16), we obtain the evolution equation for the generalized Elsässer variables as follows:

$$\frac{\partial}{\partial t} \hat{Z}(\vec{k}; t) = \sum_{\vec{p}} \sum_{\vec{q}} \frac{((\vec{k} \| \vec{p} \| \vec{q})) \lambda_{\sigma_q}^{-s_q}(q)}{(1 + \lambda_{\sigma_k}^{s_k}(k)^2)} \hat{Z}(\vec{p}; t) \hat{Z}(\vec{q}; t). \quad (39)$$

The derivation is given in Appendix 2.

5 Constants of motion from the \hat{Z} -representation perspective

Using the eigenvalue relation (25), we can rewrite the coefficients that appear in the evolution equation (39) as follows:

$$\begin{aligned} \frac{((\vec{k} \| \vec{p} \| \vec{q})) \lambda_{\sigma_q}^{-s_q}(q)}{(1 + \lambda_{\sigma_k}^{s_k}(k)^2)} &= \alpha^{-1} (\hat{k} | \hat{p} | \hat{q}) \\ &\times \frac{\lambda_{\sigma_k}^{s_k}(k) \lambda_{\sigma_p}^{s_p}(p) \lambda_{\sigma_q}^{s_q}(q) - \lambda_{\sigma_k}^{-s_k}(k) \lambda_{\sigma_p}^{-s_p}(p) \lambda_{\sigma_q}^{-s_q}(q)}{\lambda_{\sigma_k}^{s_k}(k) - \lambda_{\sigma_k}^{-s_k}(k)} \lambda_{\sigma_p}^{s_p}(p). \end{aligned} \quad (40)$$

The symmetric properties of this coefficient naturally lead to four conservation laws. Surely, it is well known that the incompressible HMHD system has three independent constants of motion, total plasma energy, magnetic helicity, and hybrid helicity (Turner 1986).

³The value of triple product $(* | * | *)$ is given by

$$(\hat{k} | \hat{p} | \hat{q}) = \frac{e^{i\Psi\{\vec{k}, \vec{p}, \vec{q}\}} |\vec{p} \times \vec{q}|}{2\sqrt{2}kpq} (\sigma_k k + \sigma_p p + \sigma_q q) \delta_{\vec{k} + \vec{p} + \vec{q}, \vec{0}}, \quad (36)$$

(Waleffe 1992, Galtier 2006), where the explicit expression of phase factor is

$$e^{i\Psi\{\vec{k}, \vec{p}, \vec{q}\}} = \frac{ikpq}{|\vec{p} \times \vec{q}|^3} (\vec{k} \cdot \mathbf{h}(\vec{p}, \sigma_p)) (\vec{p} \cdot \mathbf{h}(\vec{q}, \sigma_q)) (\vec{q} \cdot \mathbf{h}(\vec{k}, \sigma_k)). \quad (37)$$

The first two are derived from the skew symmetry of the coefficient $((\tilde{k} \parallel \tilde{p} \parallel \tilde{q}))$. The two identities given by

$$\sum_{\tilde{k}} \sum_{\tilde{p}} \sum_{\tilde{q}}^{(\tilde{k} + \tilde{p} + \tilde{q} = \vec{0})} ((\tilde{k} \parallel \tilde{p} \parallel \tilde{q})) \lambda_{\sigma_q}^{-s_q}(q) \hat{Z}(\tilde{k}; t) \hat{Z}(\tilde{p}; t) \hat{Z}(\tilde{q}; t) = 0, \quad (41)$$

$$\sum_{\tilde{k}} \sum_{\tilde{p}} \sum_{\tilde{q}}^{(\tilde{k} + \tilde{p} + \tilde{q} = \vec{0})} ((\tilde{k} \parallel \tilde{p} \parallel \tilde{q})) \lambda_{\sigma_k}^{-s_k}(k) \lambda_{\sigma_q}^{-s_q}(q) \hat{Z}(\tilde{k}; t) \hat{Z}(\tilde{p}; t) \hat{Z}(\tilde{q}; t) = 0. \quad (42)$$

lead to the following constants of motion⁴:

$$C_1 = \sum_{\tilde{k}} (1 + \lambda_{\sigma_k}^{s_k}(k)^2) |\hat{Z}(\tilde{k}; t)|^2, \quad (43)$$

$$C_2 = \sum_{\tilde{k}} (\lambda_{\sigma_k}^{-s_k}(k) - \lambda_{\sigma_k}^{s_k}(k)) |\hat{Z}(\tilde{k}; t)|^2. \quad (44)$$

Note that the constant C_1 is twice the total plasma energy $\int (|\mathbf{u}|^2 + |\mathbf{b}|^2) d^3\vec{x}$.

Derivations of the other two constants of motion use the symmetry of the fraction part of (40). The evolution equation of $\lambda_{\sigma_k}^+(k) \hat{Z}(\vec{k}, \sigma_k, +; t) + \lambda_{\sigma_k}^-(k) \hat{Z}(\vec{k}, \sigma_k, -; t)$ becomes

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\lambda_{\sigma_k}^+(k) \overline{\hat{Z}(\vec{k}, \sigma_k, +; t)} + \lambda_{\sigma_k}^-(k) \overline{\hat{Z}(\vec{k}, \sigma_k, -; t)} \right) \\ &= \sum_{\tilde{p}} \sum_{\tilde{q}}^{(\tilde{k} + \tilde{p} + \tilde{q} = \vec{0})} \frac{(\hat{k} \parallel \hat{p} \parallel \hat{q})}{\alpha} \left(\frac{\lambda_{\sigma_k}^+(k)^2 \lambda_{\sigma_p}^{s_p}(p) \lambda_{\sigma_q}^{s_q}(q) - \lambda_{\sigma_p}^{-s_p}(p) \lambda_{\sigma_q}^{-s_q}(q)}{\lambda_{\sigma_k}^+(k) - \lambda_{\sigma_k}^-(k)} \right. \\ & \quad \left. + \frac{\lambda_{\sigma_k}^-(k)^2 \lambda_{\sigma_p}^{s_p}(p) \lambda_{\sigma_q}^{s_q}(q) - \lambda_{\sigma_p}^{-s_p}(p) \lambda_{\sigma_q}^{-s_q}(q)}{\lambda_{\sigma_k}^-(k) - \lambda_{\sigma_k}^+(k)} \right) \lambda_{\sigma_p}^{s_p}(p) \hat{Z}(\tilde{p}; t) \hat{Z}(\tilde{q}; t) \\ &= 2\pi\sigma_k k \sum_{\tilde{p}} \sum_{\tilde{q}}^{(\tilde{k} + \tilde{p} + \tilde{q} = \vec{0})} (\hat{k} \parallel \hat{p} \parallel \hat{q}) \lambda_{\sigma_p}^{s_p}(p)^2 \lambda_{\sigma_q}^{s_q}(q) \hat{Z}(\tilde{p}; t) \hat{Z}(\tilde{q}; t) \end{aligned} \quad (45)$$

Since $\sum_{\tilde{q}} \lambda_{\sigma_q}^{s_q}(q) \hat{Z}(\tilde{q}; t) \phi(\tilde{q}; \vec{x}) = \mathbf{b}$ and $\sum_{\tilde{p}} \lambda_{\sigma_p}^{s_p}(p)^2 \hat{Z}(\tilde{p}; t) \phi(\tilde{p}; \vec{x}) = \mathbf{u} - \alpha \mathbf{j}$, the last line is the \hat{Z} -representation of the evolution of the magnetic field (11). The skew symmetry of $(\hat{k} \parallel \hat{p} \parallel \hat{q})$ between \hat{k} and \hat{q} leads to the identity

$$\sum_{\tilde{k}} \sum_{\tilde{p}} \sum_{\tilde{q}}^{(\tilde{k} + \tilde{p} + \tilde{q} = \vec{0})} (\hat{k} \parallel \hat{p} \parallel \hat{q}) \lambda_{\sigma_k}^{s_k}(k) \lambda_{\sigma_p}^{s_p}(p)^2 \lambda_{\sigma_q}^{s_q}(q) \hat{Z}(\tilde{k}; t) \hat{Z}(\tilde{p}; t) \hat{Z}(\tilde{q}; t) = 0$$

⁴The problem of the regularity of the \vec{Z} -variables that guarantee the convergence of these identities, which was discussed for the Euler equation case by Eyink, Constantin et al. (Eyink 1994, Constantin, E and Titi 1994), may remain, but we assume here appropriate regularity (cf. Chae Degond Liu 2014).

which yields the third constant of motion given by

$$C_3 = \sum_{\vec{k}, \sigma_k} \frac{\left| \sum_{s_k} \lambda_{\sigma_k}^{s_k}(k) \widehat{Z}(\vec{k}; t) \right|^2}{2\pi \sigma_k k} = \int_{\vec{x} \in M} \mathbf{b} \cdot ((\nabla \times)^{-1} \mathbf{b}) d^3 \vec{x}, \quad (46)$$

which is the magnetic helicity.

The evolution equation of $\lambda_{\sigma_k}^-(k) \widehat{Z}(\vec{k}, \sigma_k, +; t) + \lambda_{\sigma_k}^+(k) \widehat{Z}(\vec{k}, \sigma_k, -; t)$ then becomes

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\lambda_{\sigma_k}^-(k) \widehat{Z}(\vec{k}, \sigma_k, +; t) + \lambda_{\sigma_k}^+(k) \widehat{Z}(\vec{k}, \sigma_k, -; t) \right) \\ &= \sum_{\vec{p}} \sum_{\vec{q}}^{\vec{k} + \vec{p} + \vec{q} = \vec{0}} \frac{(\hat{k}|\hat{p}|\hat{q})}{\alpha} \left(\frac{-\lambda_{\sigma_p}^{s_p}(p) \lambda_{\sigma_q}^{s_q}(q) + \lambda_{\sigma_k}^-(k)^2 \lambda_{\sigma_p}^{-s_p}(p) \lambda_{\sigma_q}^{-s_q}(q)}{\lambda_{\sigma_k}^+(k) - \lambda_{\sigma_k}^-(k)} \right. \\ & \quad \left. + \frac{-\lambda_{\sigma_p}^{s_p}(p) \lambda_{\sigma_q}^{s_q}(q) + \lambda_{\sigma_k}^+(k)^2 \lambda_{\sigma_p}^{-s_p}(p) \lambda_{\sigma_q}^{-s_q}(q)}{\lambda_{\sigma_k}^-(k) - \lambda_{\sigma_k}^+(k)} \right) \lambda_{\sigma_p}^{s_p}(p) \widehat{Z}(\vec{p}; t) \widehat{Z}(\vec{q}; t) \\ &= 2\pi \sigma_k k \sum_{\vec{p}} \sum_{\vec{q}}^{\vec{k} + \vec{p} + \vec{q} = \vec{0}} (\hat{k}|\hat{p}|\hat{q}) \lambda_{\sigma_q}^{-s_q}(q) \widehat{Z}(\vec{p}; t) \widehat{Z}(\vec{q}; t). \end{aligned} \quad (47)$$

Since $\sum_{\vec{p}} \widehat{Z}(\vec{p}; t) \phi(\vec{p}; \vec{x}) = \mathbf{u}$ and $\sum_{\vec{q}} \lambda_{\sigma_q}^{-s_q}(q) \widehat{Z}(\vec{q}; t) \phi(\vec{q}; \vec{x}) = \mathbf{b} + \alpha \nabla \times \mathbf{u}$, the last line is the \widehat{Z} -representation of the evolution equation $\partial_t(\mathbf{b} + \alpha \nabla \times \mathbf{u}) = \nabla \times (\mathbf{u} \times (\mathbf{b} + \alpha \nabla \times \mathbf{u}))$. Skew symmetry of $(\hat{k}|\hat{p}|\hat{q})$ between \hat{k} and \hat{q} leads to the identity

$$\sum_{\vec{k}} \sum_{\vec{p}} \sum_{\vec{q}}^{\vec{k} + \vec{p} + \vec{q} = \vec{0}} (\hat{k}|\hat{p}|\hat{q}) \lambda_{\sigma_k}^{-s_k}(k) \lambda_{\sigma_q}^{-s_q}(q) \widehat{Z}(\vec{k}; t) \widehat{Z}(\vec{p}; t) \widehat{Z}(\vec{q}; t) = 0$$

which yields the fourth constant of motion given by

$$C_4 = \sum_{\vec{k}, \sigma_k} \frac{\left| \sum_{s_k} \lambda_{\sigma_k}^{-s_k}(k) \widehat{Z}(\vec{k}; t) \right|^2}{2\pi \sigma_k k} \quad (48)$$

$$= \int_{\vec{x} \in M} (\mathbf{b} + \alpha \nabla \times \mathbf{u}) \cdot ((\nabla \times)^{-1} \mathbf{b} + \alpha \mathbf{u}) d^3 \vec{x}, \quad (49)$$

which is the hybrid helicity. The constant C_4 converges to C_3 in the limit $\alpha \rightarrow 0$.

Although these four constants of motion are derived naturally from the symmetry of coefficients of the quadratic term (40), C_2 , C_3 , and C_4 are not independent. Subtraction of C_3 from C_4 yields

$$C_4 - C_3 = \sum_{\vec{k}, \sigma_k} \frac{\left| \sum_{s_k} \lambda_{\sigma_k}^{-s_k}(k) \widehat{Z}(\vec{k}; t) \right|^2}{2\pi \sigma_k k} - \sum_{\vec{k}, \sigma_k} \frac{\left| \sum_{s_k} \lambda_{\sigma_k}^{s_k}(k) \widehat{Z}(\vec{k}; t) \right|^2}{2\pi \sigma_k k},$$

$$\begin{aligned}
&= \alpha \sum \frac{(\lambda_{\sigma_k}^-(k)^2 - \lambda_{\sigma_k}^+(k)^2) \left(|\widehat{Z}(\vec{k}, \sigma_k, +)|^2 - |\widehat{Z}(\vec{k}, \sigma_k, -)|^2 \right)}{\lambda_{\sigma_k}^+(k) + \lambda_{\sigma_k}^-(k)}, \\
&= \alpha \sum (\lambda_{\sigma_k}^-(k) - \lambda_{\sigma_k}^+(k)) \left(|\widehat{Z}(\vec{k}, \sigma_k, +)|^2 - |\widehat{Z}(\vec{k}, \sigma_k, -)|^2 \right), \\
&= \alpha C_2.
\end{aligned} \tag{50}$$

Thus, the constant C_2 is as follows:

$$C_2 = \int_{\vec{x} \in M} (2\mathbf{b} \cdot \mathbf{u} + \alpha \mathbf{u} \cdot (\nabla \times \mathbf{u})) d^3\vec{x}, \tag{51}$$

which converges to twice the cross helicity in the limit $\alpha \rightarrow 0$. We refer to this quantity as the modified cross helicity hereafter.

6 Discussion

In this study an incompressible HMHD system was examined from a Lagrangian mechanical viewpoint. The equation of motion of an incompressible HMHD medium can be derived as a dynamical system on a semidirect product of two diffeomorphism groups, whose Lie algebra is a function space of a pair of velocity and current fields. The Riemannian metric is so defined that the Lagrangian is given by the sum of the kinetic and magnetic energies. Although the Lagrangian is same as that used in Hattori to derive a standard MHD equation (Hattori 2011), the difference between his configuration space and ours leads to different dynamical systems.

To derive the evolution equation, we followed the formulation given by Vizman (Vizman 2001). When a Lie group is extended by the semidirect product method to obtain a specific dynamical system, some vector space has been taken as a secondary set and the case of the semidirect product of two groups seems rare. Note that, the model we derived includes the standard MHD equation in the limit $\alpha \rightarrow 0$ and the Euler equation for an incompressible fluid for the $\mathbf{b} = \mathbf{0}$ case.

The Riemannian metric and commutator can be defined on the function space of a pair of velocity and magnetic fields. So the function space constitutes a Lie algebra and works as another configuration space of an HMHD system. Thus, the generalized Elsässer variables are shown to lead to remarkably simple expressions for the components of the Riemannian metric, structure constants of the Lie algebra, and the equation of motion.

Comparisons between the HMHD system and the incompressible Euler equation system should be made, since this provides an insight into the nature of the conservation laws. As had been shown by Arnold (Arnold 1966), the Euler equation for an incompressible fluid

$$\dot{\mathbf{u}} = (\mathbf{u} \times (\nabla \times \mathbf{u}))_{\mathcal{S}}$$

is obtained as a dynamical system on $S\text{Diff}(M)$ on which the Riemannian metric and Lie bracket are given by

$$\langle \mathbf{u}_1 | \mathbf{u}_2 \rangle_e := \int_{\vec{x} \in M} \mathbf{u}_1 \cdot \mathbf{u}_2 \, d^3 \vec{x}, \quad [\mathbf{u}_1, \mathbf{u}_2] = \nabla \times (\mathbf{u}_1 \times \mathbf{u}_2),$$

respectively. Since the velocity field is divergence-free, it can be expanded in CHW modes as $\mathbf{u}(\vec{x}, t) = \sum_{\hat{k}} \hat{u}(\hat{k}; t) \phi(\hat{k}; \vec{x})$, and thus, the Euler equation in CHW representation is given by

$$\frac{\partial}{\partial t} \overline{\hat{u}(\hat{k}; t)} = \sum_{\vec{p}} \sum_{\vec{q}}^{\vec{k} + \vec{p} + \vec{q} = \vec{0}} (\hat{k} | \hat{p} | \hat{q}) \sigma_q q \hat{u}(\hat{p}; t) \hat{u}(\hat{q}; t). \quad (52)$$

Both equations (39) and (52) formally obey the geodesic equation $\ddot{x}^i(t) = \Gamma_{jk}^i \dot{x}^j(t) \dot{x}^k(t)$. This is a consequence of the fact that they are expressed as a dynamical system on a specific Lie group. Formal correspondence of mathematical objects between these two systems is summarized in Table.1. For the HMHD case, the eigenvalue of the linear wave mode is given by a function of wavenumber other than a polynomial, i.e. its multiplication to eigenmode in wavenumber space is an operation of a pseudodifferential operator.

Formal correspondence of these two systems helps us to consider the mathematical foundation of the conservation laws derived in section 5. As was discussed, the symmetry of the coefficient of \hat{Z} -representation of an HMHD system derives *four* conservation laws, while two constants of motion, energy and helicity are known to be intrinsic in the Euler equation case (Khesin and Chekanov 1989).

Conservation of energy reflects the right-invariance of the Riemannian metric in each system.

Since the Euler system counterpart of the derivation process of the modified cross helicity (42) is

$$\sum_{\hat{k}} \sum_{\vec{p}} \sum_{\vec{q}}^{\vec{k} + \vec{p} + \vec{q} = \vec{0}} (\hat{k} | \hat{p} | \hat{q}) \sigma_k k \sigma_q q \hat{u}(\hat{k}; t) \hat{u}(\hat{p}; t) \hat{u}(\hat{q}; t) = 0,$$

the modified cross helicity (51) corresponds to the helicity in the Euler equation system, which is given by $H = \sum_{\hat{k}} \sigma_k k |\hat{u}(\hat{k}; t)|^2 = \int \mathbf{u} \cdot (\nabla \times \mathbf{u}) \, d^3 \vec{x}$. In the standard MHD limit ($\alpha \rightarrow 0$), the modified cross helicity remains an $O(\alpha^0)$ quantity and converges to the cross helicity.

Thus, energy and the modified cross helicity have their counterparts in the Euler equation system. Furthermore, the derivation processes have their counterparts even in the freely spinning top case, which is described by the Euler equation for a rigid body, and thus, these two constants appear to be more fundamental than the magnetic and hybrid helicities.

Magnetic and hybrid helicities are expected to be intrinsic for HMHD systems, since their derivation relies on the symmetry of the eigenvalue part of

Table 1: Correspondence of mathematical objects between the Euler equation and the HMHD equation.

object	Euler eq.	Hall MHD system
basic variables	$\mathbf{u}(\vec{x}, t)$	$\vec{\mathbf{V}}(\vec{x}, t) = \begin{pmatrix} \mathbf{u}(\vec{x}, t) \\ -\alpha \mathbf{j}(\vec{x}, t) \end{pmatrix}$ or $\vec{\mathbf{Z}}(\vec{x}, t) = \begin{pmatrix} \mathbf{u}(\vec{x}, t) \\ \mathbf{b}(\vec{x}, t) \end{pmatrix}$
intrinsic linear operator for normal modes	$\nabla \times$	$\begin{pmatrix} O & I \\ I & -\alpha \nabla \times \end{pmatrix}$
eigenvalue	$2\pi\sigma k$	$\lambda_{\sigma_k}^{s_k}(k) = \sigma_k \left(s_k \sqrt{(\pi\alpha k)^2 + 1} - \pi\alpha k \right)$
eigenfunction, normal mode	complex helical wave: $\phi(\vec{k}, \sigma_k) = \frac{\mathbf{e}_\theta(\vec{k}) + i\sigma_k \mathbf{e}_\phi(\vec{k})}{\sqrt{2}} e^{2\pi i \vec{k} \cdot \vec{x}}$	generalized Elsässer variable: $\vec{\Phi}(\vec{k}, \sigma_k, s_k) = \begin{pmatrix} \phi(\vec{k}, \sigma_k; \vec{x}) \\ \lambda_{\sigma_k}^{s_k}(k) \phi(\vec{k}, \sigma_k; \vec{x}) \end{pmatrix}$
mode indices	$\begin{cases} \vec{k} : & \text{wavenumber,} \\ \sigma : & \text{helicity,} \end{cases}$	$\begin{cases} \vec{k} : & \text{wavenumber,} \\ \sigma : & \text{helicity,} \\ s : & \text{polarity,} \end{cases}$
Riemannian metric tensor	$\delta_{\sigma_k \sigma_p} \delta_{k_x p_x} \delta_{k_y p_y} \delta_{k_z p_z}$	$(1 + (\lambda_{\sigma_k}^{s_k}(k))^2) \times \delta_{\sigma_k \sigma_p} \delta_{s_k s_p} \delta_{k_x p_x} \delta_{k_y p_y} \delta_{k_z p_z}$
structure constant of Lie algebra	$(\vec{k}, \sigma_k \vec{p}, \sigma_p \vec{q}, \sigma_q) \sigma_k k$	$\frac{((\vec{k}, \sigma_k, s_k \vec{p}, \sigma_p, s_p \vec{q}, \sigma_q, s_q)) \lambda_{\sigma_k}^{-s_k}(k)}{1 + \lambda_{\sigma_k}^{s_k}(k)^2}$

structure constant (40). This seems to reflect some unknown symmetry of an HMHD system such that the difference between the hybrid and magnetic helicities is proportional to the product of the Hall term coefficient and the modified cross helicity, which vanishes in standard MHD limit. Although Noether's theory states that some conservation laws reflect the symmetries that a dynamical system intrinsically has, investigation of the symmetry behind these two constants remains a subject for future work.

Despite the predominance of the studies from Hamiltonian mechanics viewpoint, we took a Lagrangian mechanics approach, i.e., variational formulation approach to the HMHD dynamics in the present study. The advantage of this approach is that it provides a differential geometrical framework to the predictability problem (Arnold 1966 section 11). That is, the second variation of the action is known to yield the Jacobi equation, which provides information

on the stability of paths in the configuration space (Arnold 1989 appendix 1). From this viewpoint, attraction (stable) or repulsion (unstable) of infinitesimally close paths are determined in terms of the *Riemannian curvature* associated with appropriate *affine connection* on which the time evolution is described as a *geodesic*. The value of the curvature is determined by the snapshot pair of a (possibly time dependent) solution and perturbation, i.e., the Jacobi field exerted on it without solving the evolution equation or eigenvalue problem. Thus, the Lagrangian mechanical approach is the first step to the predictability problem and we elucidated here the basic differential geometrical, Lie algebraic feature of configuration space of an HMHD system. We are now working on our planned next article on this basis.

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Appendix 1 Derivation of the value of Lie bracket in \hat{Z} -representation

Since CHW $\phi(\vec{k}; \vec{x})$ is an eigenfunction of the curl operator and the corresponding eigenvalue is $2\pi\sigma_k k$, the curl operation on CHW can be expressed in terms of eigenvalues of GEV as

$$\nabla \times \phi(\vec{k}, \sigma_k; \vec{x}) = -\alpha^{-1}(\lambda_{\sigma_k}^{s_k}(k) + \lambda_{\sigma_k}^{-s_k}(k))\phi(\vec{k}, \sigma_k; \vec{x}). \quad (53)$$

Substituting the CHW representation of the generalized Elsässer variables (20) into the Lie bracket in \vec{Z} -variable space (12), taking the inner product

defined by (13), and using the eigenvalue relations (24), (25) and (53) recursively, we obtain the following formula:

$$\begin{aligned}
\langle \vec{\Phi}(\tilde{k}) | \{ \vec{\Phi}(\tilde{p}), \vec{\Phi}(\tilde{q}) \} \rangle_Z &= \int_{\vec{x} \in M} d^3 \vec{x} \left[\phi(\hat{k}; \vec{x}) \cdot \nabla \times \left(\phi(\hat{p}; \vec{x}) \times \phi(\hat{q}; \vec{x}) \right) \right. \\
&\quad + \lambda_{\sigma_k}^{s_k}(k) \phi(\hat{k}; \vec{x}) \cdot \left(\phi(\hat{p}; \vec{x}) \times (\lambda_{\sigma_q}^{s_q}(q) \nabla \times \phi(\hat{q}; \vec{x})) + (\lambda_{\sigma_p}^{s_p}(p) \nabla \times \phi(\hat{p}; \vec{x})) \times \phi(\hat{q}; \vec{x}) \right. \\
&\quad \left. \left. - \alpha (\lambda_{\sigma_p}^{s_p}(p) \nabla \times \phi(\hat{p}; \vec{x})) \times (\lambda_{\sigma_q}^{s_q}(q) \nabla \times \phi(\hat{q}; \vec{x})) \right) \right] \\
&= (\hat{k} | \hat{p} | \hat{q}) \left[- \frac{\lambda_{\sigma_k}^{s_k}(k) + \lambda_{\sigma_k}^{-s_k}(k)}{\alpha} + \lambda_{\sigma_k}^{s_k}(k) \left(- \lambda_{\sigma_q}^{s_q}(q) \frac{\lambda_{\sigma_q}^{s_q}(q) + \lambda_{\sigma_q}^{-s_q}(q)}{\alpha} \right. \right. \\
&\quad \left. \left. - \lambda_{\sigma_p}^{s_p}(p) \frac{\lambda_{\sigma_p}^{s_p}(p) + \lambda_{\sigma_p}^{-s_p}(p)}{\alpha} - \alpha \lambda_{\sigma_p}^{s_p}(p) \lambda_{\sigma_q}^{s_q}(q) \frac{\lambda_{\sigma_p}^{s_p}(p) + \lambda_{\sigma_p}^{-s_p}(p)}{\alpha} \frac{\lambda_{\sigma_q}^{s_q}(q) + \lambda_{\sigma_q}^{-s_q}(q)}{\alpha} \right) \right] \\
&= \frac{(\hat{k} | \hat{p} | \hat{q})}{\alpha} \left(\lambda_{\sigma_k}^{s_k}(k)^2 \lambda_{\sigma_p}^{s_p}(p)^2 \lambda_{\sigma_q}^{s_q}(q)^2 - 1 \right) \lambda_{\sigma_k}^{-s_k}(k), \tag{54}
\end{aligned}$$

where $(* | * | *)$ is defined by (35).

Appendix 2 Derivation of equation of motion in \widehat{Z} -representation

Integrating by parts the first variation of action in \vec{Z} -variable space (16), we obtain the extremal condition

$$\int_0^1 dt \left[- \langle \vec{Z} | \dot{\vec{\zeta}} \rangle_Z + \langle \vec{Z} | \{ \vec{\zeta}, \vec{Z} \} \rangle_Z \right] = 0 \tag{55}$$

Substituting (27) and setting $\vec{\zeta} = \vec{\Phi}(\tilde{k})$, we obtain the Euler-Lagrange equation including the variation in the $\vec{\Phi}(\tilde{k})$ -direction

$$\left\langle \sum_{\tilde{q}} \partial_t \widehat{Z}(\tilde{q}) \vec{\Phi}(\tilde{q}) \middle| \vec{\Phi}(\tilde{k}) \right\rangle_Z = \left\langle \sum_{\tilde{q}} \widehat{Z}(\tilde{q}) \vec{\Phi}(\tilde{q}) \middle| \left\{ \vec{\Phi}(\tilde{k}), \sum_{\tilde{p}} \widehat{Z}(\tilde{p}) \vec{\Phi}(\tilde{p}) \right\} \right\rangle_Z. \tag{56}$$

Substituting (26) and (33), we obtain the evolution equation of an HMHD system for \widehat{Z} -representation

$$\begin{aligned}
&(1 + \lambda_{\sigma_k}^{s_k}(k)^2) \partial_t \widehat{Z}(\tilde{k}) \\
&= \sum_{\tilde{q}} \sum_{\tilde{p}} \frac{(\hat{q} | \hat{k} | \hat{p})}{\alpha} \left(\lambda_{\sigma_q}^{s_q}(q)^2 \lambda_{\sigma_k}^{s_k}(k)^2 \lambda_{\sigma_p}^{s_p}(p)^2 - 1 \right) \lambda_{\sigma_q}^{-s_q}(q) \widehat{Z}(\tilde{p}) \widehat{Z}(\tilde{q}). \tag{57}
\end{aligned}$$

Cyclic symmetry of $(* | * | *)$ leads to (39).